It happens that the two infima are the same. Finding $\inf X^{-1}$ subject to $X \leqslant Y$ is equivalent to finding the maximum of $\Sigma(\cos\phi_i - s)^2$ subject to $a^2\Sigma(\cos\phi_i - s)^2 \leqslant b^2\Sigma(\sin\phi_i - t)^2$. Adding the condition $\Sigma(\cos\phi_i - s)^2 + \Sigma(\sin\phi_i - t)^2 \leqslant n$, we find that $\max\Sigma(\cos\phi_i - s)^2 = nb^2/(a^2+b^2)$ and the maximum is reached if and only if (2·8) is satisfied.

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A new class of resolvable incomplete block designs

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SUMMARY

This paper describes an algorithm for constructing resolvable incomplete block designs for any number of varieties v and block size k such that v is a multiple of k. These designs are called α-designs. They include as special cases some lattice and resolvable cyclic designs. Additional designs with two block sizes differing by one plot are derived by omitting one or more varieties of the α-designs. The designs are shown to be available with high efficiency factors for a wide range of parameter values.

Some key words: Concurrence matrix; Cyclic design; Efficiency factor; Incomplete block design; Rectangular lattice; Resolvability; Square lattice; Variety trial.

1. INTRODUCTION

Every year, several hundred designs are required in the United Kingdom for statutory field trials of agricultural crop varieties. Numbers of varieties in these trials are fixed, i.e. not at the choice of the statistician, and large enough to require use of incomplete block designs or some other method of controlling error. Numbers of replications are also fixed. Designs must be resolvable, i.e. the blocks must be capable of arrangement in complete replications.

Attempts to create a file of designs for this purpose have revealed a shortage of tabulated resolvable incomplete block designs. At present the main sources of resolvable designs in the literature are (a) the lattice designs introduced by Yates (1936) specifically for variety trials, (b) the two-replicate designs described by Bose & Nair (1962) and (c) the resolvable cyclic designs considered by David (1967).

Lattice designs are available only for limited numbers of varieties and block sizes. Thus, simple and triple square lattices require v, the number of varieties, to be the square of s, the number of blocks in each replication; the block size k is also s. Further conditions are imposed on v in quadruple and higher order lattices.

Harshbarger (1949) extended the lattice principle to simple and triple rectangular lattices with v of the form $s(s-1)$ and $k = s-1$. Kempthorne (1952, Chapter 25) pointed out that similar designs are available for other v of the form $s(s-l)$ with $k = s-l$ and $s > l$ but gave few details. For plans for square and rectangular lattice designs, see Cochran & Cox (1957, Chapter 10).

Bose & Nair's (1962) designs very usefully augment the simple square and rectangular lattices but there appears to have been no parallel development for higher order lattices. David's (1967) construction for cyclic designs is capable of producing a large number of resolvable designs but again there are restrictions; this time k must equal either r, the number of replications, or a multiple of r.

Some other incomplete block designs are also resolvable. Clatworthy (1973) provided information on the resolvability or otherwise of most of the partially balanced incomplete block designs with two associate classes in his extensive tables. The resolvable designs are, however, usually either square lattices or less efficient alternatives.

In the present paper we describe a method for constructing a class of resolvable equiblock-sized designs, called α-designs, with no limitation on block size other than the unavoidable constraint that k must be a factor of v. This method has been developed to provide a simple computer algorithm for automatic production of plans for variety trials. We also consider the provision of designs for numbers v without a factor in the range of acceptable block sizes.

2. Conditions of the design problem

Under the conditions operating at present in the United Kingdom variety testing system numbers of trials, replications, and control varieties are specified as part of the trial system. The total number of varieties depends on the number of new varieties submitted for test, and the availability of seed. This number may vary from centre to centre and is often not known until shortly before sowing. All varieties are equally replicated.

Practical field conditions dictate that all designs used for these trials are resolvable. Thus some important disease measurements are expensive and have to be restricted to one or two replications. Again, large trials cannot always be completely drilled or harvested in a single session. Use of resolvable designs allows these operations to be done in stages, with one or more complete replications dealt with at each stage.

Yates (1939, 1940) has pointed to other advantages of resolvable designs. On page 325 of his 1940 paper he noted that

> cases will arise in which the use of ordinary randomized blocks will be more efficient than the use of incomplete blocks, whereas lattice designs can never be less efficient than ordinary randomized blocks.

This advantage of lattice designs is shared by all other resolvable incomplete block designs. Yates (1940) further stated that

> incomplete block designs which cannot be arranged in complete replications are likely to be of less value in agriculture than ordinary lattice designs. Their greatest use is likely to be found in dealing with experimental material in which the block size is definitely determined by the nature of the material.

In variety trials, of course, a wide choice of block size is open to the experimenter.

We have attempted to deal with the problem of providing incomplete block designs for statutory variety trials by creating a computer file of resolvable designs for values of v, r such that $r = 2, 3, 4$ and $v \leqslant 100$. Block sizes are chosen, in the range 4 to 16, to achieve a compromise between the reduction in within-blocks variance associated with small blocks and the loss of within-block contrasts. Equal-sized blocks of k plots with $k < v$ can be used only when v is a multiple of k. Thus we have the further conditions that for $s > 1$, $v = ks$ and $k = 4, \ldots, 16$. When v is not a multiple of a suitable value of k two block sizes are allowed; we impose the condition that the difference in size must not be more than one plot. For example, 46 varieties could be tested in a design with three blocks of 6 and four blocks of 7 per replication.

3. Construction of resolvable designs with blocks of equal size

We now describe the construction of α-designs for v varieties with s blocks of k plots in each of r replications, $v = ks$.

The construction starts with a $k \times r$ array α with elements $a(p, q)$ in the set of residues mod s $(p = 1, \ldots, k; q = 1, \ldots, r)$. Each column of α is used to generate $s-1$ further columns by cyclic substitution. The resulting $k \times rs$ array will be denoted by α^*. Next we add s to all elements in row 2 of α^*, $2s$ to all elements in row 3, and so on. The elements of the resulting

Table 1. *Construction of design for* 20 *varieties in* 3 *replications each with* 4 *blocks of* 5 $(v = 20, r = 3, s = 4, k = 5)$

Generating array, α

0	0	0
0	1	2
0	2	3
0	3	1
0	3	2

Intermediate array, α^*

0	1	2	3	0	1	2	3	0	1	2	3
0	1	2	3	1	2	3	0	2	3	0	1
0	1	2	3	2	3	0	1	3	0	1	2
0	1	2	3	3	0	1	2	1	2	3	0
0	1	2	3	3	0	1	2	2	3	0	1

Plan

Replication I				Replication II				Replication III			
1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	0	1	2	3	0	1	2	3
4	5	6	7	5	6	7	4	6	7	4	5
8	9	10	11	10	11	8	9	11	8	9	10
12	13	14	15	15	12	13	14	13	14	15	12
16	17	18	19	19	16	17	18	18	19	16	17

array are now the symbols $0, 1, \ldots, v-1$ representing the v varieties. The columns are the blocks of the required design. Each set of columns generated from the same column of α constitutes a complete replication.

For example, Table 1 gives the construction of a design for three replications of 20 varieties, each replication consisting of four blocks of five.

The number of concurrences of any two varieties, i.e. the number of blocks containing both varieties, can be determined by inspection of α without actually constructing the complete design. This facility is of great value in choosing suitable generators. Thus, the number of concurrences of varieties i and j is the frequency of $(j-i) \bmod s$ in the set of r differences $\{a(p_j, q) - a(p_i, q)\} \bmod s$ $(q = 1, \ldots, r)$, where p_i is one more than the integral part of i/s.

To illustrate, we consider the concurrence of varieties 7 and 8 in the example. The values of p_i, p_j and $(j-i) \bmod s$ are 2, 3 and 1. The difference 1 occurs twice in the set 0, 1, 1 of differences obtained by subtracting the elements of the second row of α from the elements of the third row. Hence varieties 7 and 8 concur twice. It follows immediately that the pairs 4 and 9, 5 and 10, and 6 and 11 also concur twice.

A design with concurrences $g_1, g_2, \ldots$ will be referred to as an $\alpha(g_1, g_2, \ldots)$-design; the example is therefore an $\alpha(0, 1, 2)$-design.

The generating array used in the example is of a particular kind, called a reduced array, with all elements zero in the first row and first column. Other arrays can be used but each design so obtained is isomorphic to the design given by a reduced array. Thus, for example, the array α_1 given by

$$\alpha_1 = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

with $s = 4$ generates a design that is isomorphic to the design of Table 1. This can be demonstrated by converting α_1 to reduced form by the following operations:

(*a*) add 3 to all elements in column 1, 1 to all elements in column 2 and 3 to all elements in column 3, in each case reducing mod 4;

(*b*) add 3 to the elements in row 5 and again reduce mod 4.

The reduced form is therefore the array in Table 1. Operation (*a*) results in a rearrangement of blocks, and operation (*b*) in a relabelling of varieties.

Other operations generating isomorphic designs include permutation of rows, permutation of columns and multiplication of all elements by an integer that is coprime to s.

4. Derived designs with unequal block sizes

Often none of the factors of v falls in the range of acceptable block sizes. We then use designs with two different block sizes. These designs exist when v can be expressed in the form $v = s_1 k_1 + s_2 k_2$, where s_1, s_2, k_1, k_2 are positive integers. Each replication consists of s_1 blocks of k_1 and s_2 blocks of k_2. To minimize the inequality in block size we impose the condition $k_2 = k_1 - 1$. The designs are derived from α-designs as follows:

(1) construct an α-design for $v + s_2$ varieties with $s = s_1 + s_2$ blocks of k_1 plots in each replication;

(2) delete a set of s_2 varieties, no two of which concur. The varieties labelled $v, \ldots, v + s_2 - 1$ provide such a set.

For example, deletion of the varieties labelled 17, 18, 19 from the design in Table 1 gives a design for 17 varieties with one block of 5 and three blocks of 4 in each of three replications.

5. Choice of designs

Efficient $\alpha(0, 1)$-designs or $\alpha(0, 1, 2)$-designs can be constructed for 376 of the 414 combinations of r, v and k satisfying the conditions of §2. Many further useful designs with two block sizes can be derived by deletion of one or more varieties of α-designs.

Tables of suitable generating arrays and full details of the theory, construction and properties of these designs are in a recent Edinburgh Ph.D. thesis by E. R. Williams. Copies of the tables are also available separately.

In the present paper we describe some of the considerations governing the choice of generating arrays, establish a few existence results and briefly compare the designs with some known resolvable designs.

In constructing the tables the aim has been to choose, for every admissible value of k, a single α-design with efficiency factor E as large as possible. Any factor of v is an admissible

value of k. The efficiency factor is defined as the ratio r'/r such that σ^2/r' is the average variance of normalized contrasts and σ^2 the error variance in the within-block analysis of the incomplete block design. Designs with maximum E among all α-designs are called α-optimal.

A combination of theory and computation has been used in constructing the tables. When $r = 2$, α-optimal designs are generated by reduced α arrays with second column given by the leading block of the most efficient symmetrical cyclic incomplete block design for s varieties in blocks of k (Williams, 1976). John, Wolock & David (1972) give many of the cyclic incomplete block designs required for this purpose. Arrays giving optimal designs are also known for a few special combinations of v, k and $r = 3$ or 4. In most cases, however, the tabulated designs have been obtained by an empirical two-stage process. Stage 1 consists of the selection of a set of nonisomorphic designs judged likely to be of high efficiency. In stage 2, E values are calculated and the final selection made.

Table 2. *Efficiency check of α-designs and derived designs with $r = 4$, $s = 5$*

v	E	E_*	E/E_*	v	E	E_*	E/E_*
18	0·7399	0·7612	0·9720	30	0·8392	0·8447	0·9935
19	0·7551	0·7714	0·9789	40	0·8765	0·8797	0·9964
20	0·7686	0·7808	0·9844	50	0·9018	0·9018	1·0000
21	0·7804	0·7895	0·9885	60	0·9160	0·9171	0·9988
22	0·7911	0·7975	0·9920	70	0·9278	0·9283	0·9995
23	0·8010	0·8049	0·9952	80	0·9364	0·9368	0·9996
24	0·8099	0·8118	0·9977	90	0·9431	0·9435	0·9996
25	0·8182	0·8182	1·0000	100	0·9489	0·9489	1·0000
26	0·8232	0·8242	0·9988				
27	0·8278	0·8298	0·9976				
28	0·8319	0·8351	0·9962				
29	0·8357	0·8400	0·9949				

Among the criteria used in stage 1 are (*a*) minimization of the range of off-diagonal elements in the concurrence matrix, and (*b*) minimization of the number of off-diagonal elements greater than one. These criteria are intuitively acceptable and (*a*), at least, has been used by other workers, e.g. John (1966). Thus, given the choice, we prefer an $\alpha(0, 1)$-design to an $\alpha(0, 1, 2)$-design. If we have to use an $\alpha(0, 1, 2)$-design we prefer one with as few pairs of varieties as possible concurring twice.

We note that the construction of α-designs prevents the concurrence of any two varieties i and j such that the integral parts of i/s and j/s are equal. Hence neither balanced α-designs nor $\alpha(1, 2)$-designs exist. Balanced designs have, however, already been ruled out in the present application by the specification in §2. The largest number of replications allowed by the specification is k; for balance the number must be at least $k + 1$.

The empirical two-stage procedure gives no assurance that a selected design is α-optimal. We can, however, often show that there is little potential for future improvement by comparing the final value of E with an upper bound given by the efficiency factor E_* that would be obtained if all contrasts confounded in any one replication were completely orthogonal to contrasts confounded in other replications. Under this usually unattainable condition the variance of a normalized contrast in any of the r confounded sets of $s - 1$ degrees of freedom would be $(r-1)^{-1}\sigma^2$, where σ^2 is the within-block error variance; the variance of a contrast not confounded in any replication would be $r^{-1}\sigma^2$. The average variance would be

$$\frac{1}{v-1}\left\{\frac{r(s-1)}{r-1}+\frac{v-1-r(s-1)}{r}\right\}\sigma^2.$$

Hence

$$E_* = \frac{(v-1)(r-1)}{(v-1)(r-1)+r(s-1)}.$$

Table 2 gives values of E, E_* and their ratios for tabulated α-designs and derived designs with five blocks in each of four replications. Two checks show that there is little scope for improvement.

(i) The ratios E/E_* are close to unity.

(ii) There is a smooth progression of values of E/E_* with changing v.

The square lattice designs for $v = 25$ and the related designs for $v = 50, 75, 100$ are optimal, i.e. with largest possible E. Almost maximum E values are achieved for all v larger than 25.

6. Availability of $\alpha(0, 1)$-designs

A necessary condition for the existence of an $\alpha(0,1)$-design with s blocks of size k in each of r replications is that $k \leqslant s$.

Proof is in two stages. First we verify that no two blocks can have two or more varieties in common. Otherwise those varieties would concur two or more times and the design would not be $(0,1)$. Now consider one particular block B. This block must have a single variety in common with exactly $k(r-1)$ of the $s(r-1)$ blocks that are not in the same replication as B. Hence $k(r-1) \leqslant s(r-1)$ and the condition $k \leqslant s$ follows. The condition and proof apply not only to $\alpha(0,1)$-designs but to any binary resolvable designs with numbers of concurrences 0 or 1.

The further necessary condition that $r \leqslant s$ is automatically satisfied in the present application since the largest r is not larger than the smallest k; this condition would, however, become operative if r were increased beyond 4.

Although necessary, the condition $k \leqslant s$ is not sufficient to guarantee the existence of an $\alpha(0,1)$-design. However, $\alpha(0,1)$-designs are available for 188 of the 198 combinations of r, v and k satisfying the restrictions imposed in §2 and $k \leqslant s$. The exceptions are listed in Table 4(ii). Known designs include the following general series:

I. $r = 2$, $k \leqslant s$;
II. $r = 3$, s odd, $k \leqslant s$;
III. $r = 3$, s even, $k \leqslant s-1$;
IV. $r = 4$, s odd but not a multiple of 3, $k \leqslant s$.

No $\alpha(0,1)$-designs are known with $r = 3$, s even and $k = s$. Series IV provides about half of the required designs with four replications. No general solutions have been found with s even or a multiple of 3 but individual solutions are available for all combinations within the required range except those listed in Table 4(ii).

Generating arrays for the four series are in Table 3. Arrays for smaller k can be obtained by deleting appropriate numbers of rows. It is important to note, however, that the resulting designs are not necessarily α-optimal.

Table 3. *Generators for $\alpha(0,1)$-designs of series I–IV*

Series I $r = 2, k \leqslant s$		Series II $r = 3$, s odd, $k \leqslant s$			Series III $r = 3$, s even, $k \leqslant s-1$			Series IV $r = 4$, s odd, $s \not\equiv 0 \bmod 3$, $k \leqslant s$			
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	$s-1$	0	1	s'	0	1	$s-1$	s''
0	2	0	2	$s-2$	0	2	1	0	2	$s-2$	1
⋮	⋮	⋮	⋮	⋮	0	3	$s'+1$	0	3	$s-3$	$s''+1$
0	$s-1$	0	$s-1$	1	0	4	2	0	4	$s-4$	2
					⋮	⋮	⋮	⋮	⋮	⋮	⋮
					0	$s-3$	$s-2$	0	$s-3$	3	$s''-2$
					0	$s-2$	$s'-1$	0	$s-2$	2	$s-1$
								0	$s-1$	1	$s''-1$
					$s' = \frac{1}{2}s$			$s'' = \frac{1}{2}(s+1)$			

7. Availability of $\alpha(0, 1, 2)$-designs

Almost all the gaps left by $\alpha(0,1)$-designs are filled by α-designs with some pairs of varieties concurring twice. These designs, like all other α-designs, must include some zero concurrences; usually, but not always, there are also single concurrences.

When r is 2 no pair of varieties can concur more than twice. When r is 3 or 4, $\alpha(0,1,2)$-designs exist if and only if $k \leqslant s^2$. A further condition $r \leqslant 2s$ is also necessary but does not affect availability until $r > 4$.

The sufficiency of the condition $k \leqslant s^2$ can be demonstrated by constructing an $s^2 \times 4$ array with zero elements in column 1 and elements in columns 2, 3 and 4 given by the rows, columns and letters of any $s \times s$ Latin square. The array generates a four-replicate $\alpha(0,1,2)$-design with $k = s^2$. Designs for smaller k are obtained by omitting one or more rows of the array and designs for $r = 3$ by omitting the fourth column. Necessity is proved by showing that an array with more than s^2 rows gives a design with at least one pair of varieties concurring three or more times. Columns 2 and 3 of this array must be such that $a(p_i, 2) = a(p_j, 2)$ and $a(p_i, 3) = a(p_j, 3)$ for at least one pair of rows p_i, p_j $(p_i \neq p_j)$. Then

$$a(p_i, q) - a(p_j, q) = 0 \bmod s$$

for $q = 1, 2$ and 3; that is, varieties $(p_i - 1)s$ and $(p_j - 1)s$ concur at least three times.

The condition $k \leqslant s^2$ rules out the possibility of three- and four-replicate $\alpha(0,1,2)$-designs for (i) even v, $32 \geqslant v \geqslant 10$, $k = \frac{1}{2}v$, and (ii) $v = 0 \bmod 3$, $48 \geqslant v \geqslant 30$, $k = \frac{1}{3}v$. All other combinations of parameters in the range specified in §2 are covered by $\alpha(0,1)$-designs and $\alpha(0,1,2)$-designs.

8. Comparison with lattice designs

The four series of $\alpha(0,1)$-designs described in §6 include some but not all square and rectangular lattices. The following lattice designs with parameters in the range specified in §2 are not α-designs:

A. rectangular lattices for $r = 3$ or 4, s even;
B. square lattices for $r = 3$ or 4, s even;
C. square and rectangular lattices for $r = 4$, $s = 9$.

Additional design-generating algorithms would be required to include these lattice designs in the file. Efficient substitute designs are, however, available in the α-series. Table 4 shows that we can use suitably chosen $\alpha(0,1)$-designs instead of lattice designs A with trivial loss of efficiency, and that $\alpha(0,1,2)$-designs can be used as efficient substitutes for lattice designs B and C; no $\alpha(0,1)$-designs are available for the latter.

Table 4 includes one set of parameters ($r = 4, v = 30, k = 5$) for which an $\alpha(0,1)$-design exists but no rectangular lattice. The value $E = 0{\cdot}8046$ for the α-design compares well with the upper limit 0·8131 given by E_*.

Table 4. *Efficiency factors for $\alpha(0,1)$ and $\alpha(0,1,2)$-designs used instead of square and rectangular lattices*

(i) $\alpha(0, 1)$-design

r	v	k	$\alpha(0, 1)$-design	Lattice design
3	30	5	0·7843	0·7856
	56	7	0·8355	0·8358
	90	9	0·8661	0·8663
4	30	5	0·8046	*
	56	7	0·8515	0·8518
	90	9	0·8794	0·8796

(ii) $\alpha(0, 1, 2)$-design

r	v	k	$\alpha(0, 1, 2)$-design	Lattice design
3	16	4	0·7538	0·7692
	36	6	0·8186	0·8235
	64	8	0·8549	0·8571
	100	10	0·8788	0·8800
4	16	4	0·7770	0·7895
	36	6	0·8360	*
	64	8	0·8692	0·8710
	72	8	0·8599	0·8672
	81	9	0·8759	0·8824
	100	10	0·8909	0·8919

* Design nonexistent.

Similarly an $\alpha(0,1,2)$-design can be used effectively to fill the gap left by the nonexistence of a square lattice design for four replications of 36 varieties. The efficiency factor of the α-design is 0·8360; the upper bound is 0·84.

The array

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 2 & 5 & 4 \\ 0 & 3 & 1 & 2 \\ 0 & 4 & 3 & 1 \\ 0 & 5 & 4 & 0 \end{bmatrix}$$

generates the $\alpha(0,1,2)$-design; the first five rows generate the $\alpha(0,1)$-design for 30 varieties.

9. Resolvable cyclic designs

David (1967) has shown that a symmetrical cyclic incomplete block design with initial block

$$(d_0 k, d_1 k + 1, d_2 k + 2, \ldots, d_{k-1} k + k - 1)$$

is resolvable; the numbers $d_0, d_1, \ldots, d_{k-1}$ are residues mod s. This design is also an α-design

with generating array given by the matrix

$$\begin{bmatrix} d_0 & d_{k-1}+1 & \ldots & d_1+1 \\ d_1 & d_0 & \ldots & d_2+1 \\ d_2 & d_1 & \ldots & d_3+1 \\ \vdots & \vdots & & \vdots \\ d_{k-1} & d_{k-2} & \ldots & d_0 \end{bmatrix},$$

with all elements reduced mod s.

For example, design C53 given by John *et al.* (1972) for $v = 24$, $r = k = 4$ has leading block 0, 1, 10, 3 and is therefore an α-design with generating array in original and reduced form

$$\begin{bmatrix} 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 4 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$

In a more general series of resolvable cyclic designs for integral k/r the leading block consists of the k possible sums of one element of the set S_1 and one element of the set S_2, where

$$S_1 = \{0, b, 2b, \ldots, (k-r)b/r\},$$
$$S_2 = \{d_0 r, d_1 r + 1, d_2 r + 2, \ldots, d_{r-1} r + r - 1\},$$

and b is the total number of blocks, that is rs. These designs also are isomorphic to α-designs.

This series can be regarded as a subset of the class of α-designs with constraints on the number and contents of the columns of the generating arrays. John *et al.* (1972) include a wide range of designs of this type for $r = 2$ and a few for $r = 3, 4$.

10. Availability of derived designs with unequal block sizes

The availability of designs derived by the method described in §4 is determined by the existence of solutions to the equation $v = s_1 k_1 + s_2 k_2$. Solutions exist for many but not all combinations of v and k_1. They can be obtained systematically for a given combination by determining the smallest positive integer d such that $v - dk_2 = ck_1$, where c is also a positive integer. Then the complete set of solutions is given by

$$s_1 = c - fk_2, \quad s_2 = d + fk_1 \quad (f = 0, 1, \ldots, m),$$

where m is the largest positive integer for which mk_2 is smaller than c.

For example, if $v = 86$, $k_1 = 8$ we find $c = 9$, $d = 2$ and hence $m = 1$. There are therefore two solutions as follows: (i) $s_1 = 9, s_2 = 2$; (ii) $s_1 = 2, s_2 = 10$. Solution (i) is used if the preferred block size is 8, solution (ii) if the preferred block size is 7.

By contrast, no solution exists for $v = 41$, $k_1 = 8$ as none of the integers 34, 27, 20, 13, 6 is divisible by 8. The complete set of values of v for which no solution is available when $k_1 = 8$ consists of: (*a*) any $v < 15$; (*b*) 21, 28, 35, 42, 49, 56; (*c*) 16, 24, 32, 40, 48, 56; (*d*) the 10 numbers 17, 18, 19, 20, 25, 26, 27, 33, 34, 41.

Multiples of 7 or 8 are in any case best dealt with by α-designs proper. Designs in blocks of 6 or mixtures of blocks of 6 and 7 are available for all v in list (*d*) except 17; designs in blocks of 9 or mixtures of blocks of 8 and 9 are available for all v in list (d) except 19 and 20. Hence a combination of α-designs with $k = 6$, 7 and 8 and derived designs with $k_1 = 7$, 8 and 9 can be used to cover the whole range of v such that $v \geqslant 12$.

More generally, a combination of α-designs with $k = k', \ldots, k''-\delta$ and derived designs with $k_1 = k'+1, \ldots, k''$ cover the range of v such that $v \geqslant 2k'$, where k' is a positive integer, k'' is the smallest integer such that $k'' \geqslant \frac{1}{2}(3k'-1)$ and δ is 0 when k' is odd and 1 when k' is even.

11. ANALYSIS

The α-designs described in the present paper have been developed specifically for a computer-based application. For this reason computational simplicity has not been considered an important criterion in the choice of designs. We have, however, been concerned to choose designs with narrow ranges of variances for pairs of varieties with equal concurrences. This permits results to be presented with only two average variances for $\alpha(0, 1)$-designs and three for $\alpha(0, 1, 2)$-designs. This type of simplification has been used for rectangular lattices by Cochran & Cox (1957, p. 422).

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A geometric characterization of connectedness in a two-way design

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SUMMARY

A geometric characterization of connectedness in a two-way design is given. This characterization is used to study four different analyses of the usual hypotheses about row effects, column effects and interactions in two-way designs with empty cells.

Some key words: Analysis of variance; Connectedness; Orthogonal sums of squares; Two-way design; Vector space.

1. INTRODUCTION

The purpose of this work is to continue the examination of the geometry of unbalanced, two-way designs begun by Burdick, Herr, O'Fallon & O'Neill (1974). In particular the case of no observations for some of the treatment combinations is considered.

In performing an analysis of variance on a two-way design, the squared length of the projection of the data vector on a subspace, G, of the estimation space, R_T, corresponding to violations of the null hypothesis is compared with a stochastically independent estimate of the within-cell variability. Burdick *et al.* (1974) discussed several possible subspaces G corresponding to violations of the usual null hypotheses when there is at least one observation per cell. Table 1 summarizes that work for hypotheses in a 3×4 design: H_r: no difference in main row effects, H_c: no difference in column main effects, H_i: no interaction. The model used here is the cell mean model $Y_{pqr} = \mu_{pq}+e_{pqr}$ ($p = 1, 2, 3; q = 1, 2, 3, 4$; $r = 1, \ldots, n_{pq}$) or $Y = T\beta+e$ for Y the vector of Y_{pqr}, β the vector of μ_{pq}, T the design matrix and e the error vector of e_{pqr}. It is assumed that e is distributed as $N(0, \sigma^2 I)$. Also

$$n_{p.} = \Sigma_q n_{pq}, \quad n_{.q} = \Sigma_p n_{pq}, \quad n_{..} = \Sigma_p \Sigma_q n_{pq}, \quad \mu_{p.} = \tfrac{1}{4}\Sigma_q \mu_{pq},$$
$$\mu_{.q} = \tfrac{1}{3}\Sigma_p \mu_{pq}, \quad \mu_{*q} = \Sigma_p(n_{pq}/n_{.q})\,\mu_{pq}, \quad \mu_{p*} = \Sigma_q(n_{pq}/n_{p.})\,\mu_{pq}.$$

The notation for the subspaces G_r, G_c and G_i is approximately that of Burdick *et al.* (1974). To summarize briefly, the spaces A and B which are defined in §2 are converted into G's by either first orthogonalizing for the mean and then taking best estimates, denoted for example by $A\,|{*}\,J$, $B\,|{*}\,J$, or by first transforming the space using T and then adjusting, for example $\hat{A} \mid \hat{B} = T(A) \mid T(B)$. Here $\hat{A} \mid \hat{B}$ is defined to be the orthogonal complement of $\hat{B}$ in $\hat{A}+\hat{B}$, that is $\hat{A}+\hat{B} = \hat{A} \mid \hat{B} \oplus \hat{B}$. Further, J is the space spanned by the $k \times 1$ vector of all 1's and k is determined from the context. The angle θ between $\hat{A} \mid J$ and $\hat{B} \mid J$ measures the nonorthogonality of the design and in this case is in fact a pair of angles, (θ_1, θ_2). Examples of these angles appear in Tables 3, 4 and 6.

From Table 1 it is clear that in the unbalanced two-way model there are trade-offs between orthogonality of G_r and G_c and convenience and/or suitability of parametric hypotheses. In choosing an appropriate analysis, I consider that careful thought should